$\boxed{\text{Lemma}}$ Suppose $f(x)$ is continuous on $\mathbb{R}$ and even. If $P.V.\int_{-\infty}^{\infty} f(x)\,dx$ exists, then the improper integral exists and

$$\int_{-\infty}^{\infty} f(x)\,dx = P.V.\int_{-\infty}^{\infty} f(x)\,dx.$$

Moreover,

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\,P.V.\int_{-\infty}^{\infty} f(x)\,dx.$$

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx \\
&= \lim_{R_1\to\infty}\frac{1}{2}\int_{-R_1}^{R_1} f(x)\,dx + \lim_{R_2\to\infty}\frac{1}{2}\int_{-R_2}^{R_2} f(x)\,dx \\
&= \frac{1}{2}\,P.V.\int_{-\infty}^{\infty} f(x)\,dx + \frac{1}{2}\,P.V.\int_{-\infty}^{\infty} f(x)\,dx \\
&= P.V.\int_{-\infty}^{\infty} f(x)\,dx.
\end{aligned}$$

Along the way, we also proved

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\,P.V.\int_{-\infty}^{\infty} f(x)\,dx.$$

Improper Integrals of Rational Functions

Assumptions:

(1) $f(x) = \frac{p(x)}{q(x)}$ is a rational function with real coefficients and such that $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros and atleast one zero with positive imaginary part.

(3) $f(x)$ is an even function.

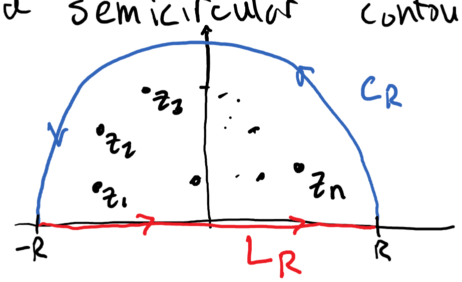
We describe a method to compute the integrals:

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx \qquad \text{and} \qquad \int_{0}^{\infty} \frac{p(x)}{q(x)}\,dx .$$

Step 1: Identify the singularities of f that lie above the real axis. By assumption, there is at least one. Label them

$$z_1, \ldots, z_n.$$

Step 2: Define a semicircular contour C as follows:



$$C = C_R + L_R$$

C_R: the semicircle parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$

L_R: the line segment joining $-R$ to R.

Choose $R > 0$ such that $R > \max_{i=1}^{n} |z_i|$.

Step 3: Apply the residue theorem:

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z).$$

Parametrize L_R via $z(x) = x$, $-R \leq x \leq R$. Then

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

Hence,

$$\text{P.V} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z) - \lim_{R\to\infty} \int_{C_R} f(z)\,dz.$$

Since $f(x)$ is even $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx$.

Step 4: Prove that $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\,dz = 0$. This can always be proved if, for instance, $\deg p(z) + 2 \leq \deg q(z)$.

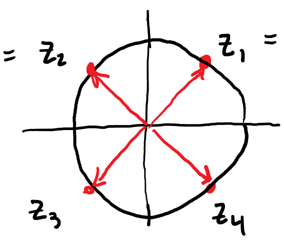
$\boxed{\text{Example}}$ Compute $\displaystyle\int_0^\infty \frac{1}{x^4+1}\,dx$.

The singularities of $f(z) = \frac{1}{z^4+1}$ are the solutions of $z^4 = -1$.

$$(-1)^{1/4} = e^{\frac{1}{4}\log(-1)} = e^{\frac{1}{4}(\ln|-1| + i\arg -1)}$$

$$= e^{\frac{1}{4}i(\pi + 2k\pi)}, \quad k = 0,1,2,3.$$

$$= e^{\frac{i\pi}{4}} \cdot e^{\frac{ik\pi}{2}}$$



Integrat $f(z)$ over the semicircular contour with $R>1$. By the residue theorem, we get

$$\int_{-R}^{R} \frac{1}{x^4+1} = 2\pi i \left(\operatorname*{Res}_{z=z_1} \frac{1}{z^4+1} + \operatorname*{Res}_{z=z_2} \frac{1}{z^4+1} \right) - \int_{C_R} \frac{1}{z^4+1}\,dz.$$

Let $p(z)=1$ and $q(z) = z^4+1$. Then p,q are both analytic at each singularity z_k, $p(z_k) = 1 \neq 0$, $q(z_k) = 0$, and $q'(z_k) = 4z_k^3 \neq 0$. Hence, each z_k is a simple poles with residue given by

$$\operatorname*{Res}_{z=z_k} \frac{1}{z^4+1} = \frac{p(z_k)}{q'(z_k)} = \frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = -\frac{z_k}{4}.$$

Next,

$$\left| \int_{C_R} \frac{1}{z^4+1}\,dz \right| \leq \pi R \cdot \max_{|z|=R} \frac{1}{|z^4+1|}$$

$$\leq \frac{\pi R}{R^4-1} \xrightarrow{R\to\infty} 0.$$

$$|z^4+1| \geq \big||z|^4 - 1\big| = |R^4-1| = R^4-1$$

Hence,

$$\int_0^{\infty} \frac{1}{x^4+1}\,dx = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$

$$= \pi i \left(\operatorname*{Res}_{z=z_1} \frac{1}{z^4+1} + \operatorname*{Res}_{z=z_2} \frac{1}{z^4+1} \right)$$

$$= -\frac{\pi i}{4} \left(z_1 + z_2 \right)$$

$$= -\frac{\pi i}{4} \left(\cancel{\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2} + \cancel{-\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2} \right) = \frac{\pi\sqrt{2}}{4}.$$

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Improper Integrals from Fourier Analysis

Assumptions :

(1) $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function with real coefficients and such that $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros and atleast one zero with positive imaginary part.

(3) $a > 0$ and $f(z) \sin az$ (or $f(z) \cos az$) is an even function.

The same method, with a slight modification, can be used to compute the integral

$$\int_{-\infty}^{\infty} f(x) \sin ax \,dx \qquad \left(\text{or} \int_{-\infty}^{\infty} f(x) \cos ax \,dx \right).$$

We use Euler's formula to write

$$\int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx = \int_{-R}^{R} f(x)e^{iax}\,dx$$

We will simply compute the RHS, take the real or imaginary part, and then take the limit.

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$\boxed{\text{Example}}$ Compute $\displaystyle\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx$. We will integrate $f(z)e^{2iz}$ where $f(z)=\frac{1}{(x^2+4)^2}$ over the semicircular contour C. Clearly, $f(z)$ has a single singularity at $z=2i$ that lies above the real axis. Assuming $R>2$, $f(z)e^{2iz}$ is analytic inside and on C except at a single point. By the residue theorem

$$\int_{-R}^{R} f(x)e^{2ix}\,dx = 2\pi i \operatorname*{Res}_{z=2i} f(z)e^{2iz} - \int_{C_R} f(z)e^{2iz}\,dz.$$

To compute the residue, define $\phi(z) = \dfrac{e^{2iz}}{(z+2i)^2}$ so that

$f(z)e^{2iz} = \dfrac{\phi(z)}{(z-2i)^2}$. Moreover, $\phi(z)$ is nonzero and analytic at $z=2i$. Hence $z=2i$ is a pole of order $m=2$ and

$$\operatorname*{Res}_{z=2i} f(z)e^{2iz} = \frac{\phi^{(2-1)}(2i)}{(2-1)!} = \phi'(2i).$$

We have

$$\phi'(2i) = \left.\frac{2ie^{2iz}(z+2i)^2 - 2(z+2i)e^{2iz}}{(z+2i)^4}\right|_{2i} = \frac{2ie^{-4}(4i)^2 - 2(4i)e^{-4}}{(4i)^4}$$

$$= \frac{2(4i)e^{-4}(4i^2-1)}{(4i)^4} = \frac{2e^{-4}(-5)}{(4i)^3} = \frac{5}{32i}e^{-4}.$$

Then we have

$$\left|\int_{C_R} f(z)\cos 2z\right| \le \left|\int_{C_R} \frac{1}{(z^2+4)^2} e^{2iz}\,dz\right| \le \pi R \max_{|z|=R} \frac{|e^{2iz}|}{|z^2+4|^2}$$

$$\le \frac{\pi R}{R^2-4} \xrightarrow{R\to\infty} 0.$$

Im $z>0$

$$|e^{2iz}| = |e^{2ix}||e^{-2y}| = |e^{-2y}| \le 1$$

$$|z^2+4| \ge |z|^2-4 = R^2-4$$

We have

$$\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{1}{2}\,\text{P.V}\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2}$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty} \text{Re}\int_{-R}^{R} f(x)e^{2ix}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\left(\frac{10\pi i}{32i}e^{-4} - \int_{C_R} f(z)e^{2iz}\,dz\right)$$

$$= \frac{5\pi}{32}e^{-4}.$$

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The preceding method works if

$$\deg p(x) + 2 \le \deg q(x).$$

If not, the triangle inequality for contour integrals may not be enough to prove

$$\lim_{R\to\infty}\int_{C_R} f(z)\,e^{aiz}\,dz = 0.$$

In this case, you may be able to use Jordans lemma instead:

$\boxed{\text{Lemma (Jordan)}}$ Assume

(1) f is analytic at all points in the upper half plane ($\operatorname{Im} z > 0$) that are exterior to some circle $|z| = R_0$.

(2) C_R is the semicircle ($z(t) = Re^{it}$ $0 \le t \le \pi$) with $R > R_0$.

(3) There exists $M_R > 0$ such that $|f(z)| \le M_R$ for all $z \in C_R$ and $\lim_{R \to \infty} M_R = 0$.

Then for any $a > 0$,

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{aiz} \, dz = 0.$$

Proof. Assume this w/out proof. See the book. $\square$

$\boxed{\text{Example}}$ Compute $\displaystyle\int_0^\infty \frac{x \sin 2x}{x^2+3} \, dx$.

Let $f(z) = \dfrac{z}{z^2+3} = \dfrac{z}{(z-\sqrt{3}i)(z+\sqrt{3}i)}$. We integrate $f(z)e^{2iz}$ over the semicircular contour C. The only singularity lying above the real axis is $z = \sqrt{3}i$.

Compute the residue: write $p(z) = ze^{2iz}$ and $q(z) = z^2+3$. Both are analytic at $z = \sqrt{3}i$, $p(\sqrt{3}i) \neq 0$, $q(\sqrt{3}i) = 0$, $q'(\sqrt{3}i) = 2\sqrt{3}i \neq 0$. So $z = \sqrt{3}i$ is a simple pole with residue

$$\operatorname*{Res}_{z=\sqrt{3}i} f(z)e^{2iz} = \frac{p(\sqrt{3}i)}{q'(\sqrt{3}i)} = \frac{1}{2} e^{-2\sqrt{3}}.$$

By the residue theorem,

$$\int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\, dx = \operatorname{Im} \int_{-R}^{R} f(x) e^{2ix} dx$$
$$= \operatorname{Im}\left(\pi i e^{-2\sqrt{3}} - \int_{C_R} f(z) e^{2iz} dz \right)$$
$$= \pi e^{-2\sqrt{3}} - \operatorname{Im} \int_{C_R} f(z) e^{2iz} dz.$$

So, we just need to show that

$$\lim_{R\to\infty} \operatorname{Im} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\, dz = 0.$$

For any $|z| = R$, we have

$$\left| \frac{z}{z^2+3} \right| = \frac{|z|}{|z^2+3|} \leq \frac{|z|}{|z|^2-3} = \frac{R}{R^2-3} =: M_R.$$

Since $\lim_{R\to\infty} M_R$, by Jordans Lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\, dz = 0$$

which implies

$$\lim_{R\to\infty} \operatorname{Im} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\, dz = 0.$$

Hence,

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\, dx = \frac{1}{2}\, \text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3}\, dx$$
$$= \frac{1}{2} \pi e^{-2\sqrt{3}}$$

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